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Gravitational collapse and frequency shift

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Abstract. This paper treats the problem of a spherically symmetric collapse in the background, not of an empty space, but of an expanding universe. Transformation formulae are presented which make the metric tensor components, together with their first derivatives, continuous at different boundaries. It is shown that collapse takes an infinite time to reach an observer in the expanding system, although it takes a finite time to reach an observer located in the collapsing system. The occurrence of the event horizon for different models is studied and the expression for frequency shift is deduced. It indicates a cut-off as the collapsing system reaches the Schwarzschild singularity.

1. Introduction

The problem of gravitational collapse has usually been studied in the background of a Schwarzschild field which asymptotically becomes Euclidean. It would be interesting and a little more realistic to consider the case where the collapsing system forms part of an expanding universe. In a recent paper Raychaudhuri (1966) has considered a spherically symmetric 'universe' in which a limited spherical region ultimately collapses, while the universe at large monotonically expands and at large distances the line element tends asymptotically to that for the isotropic universe of negative space curvature. He found that the ever-expanding region does not receive any signal from a point in the collapsing region once contraction sets in there. In Raychaudhuri's paper it is assumed that a single co-ordinate network, which is everywhere co-moving, can be introduced throughout the universe without introducing singularities. This necessitates, in particular, that the velocity field must be continuous. However, when one has coexisting collapsing and expanding regions, it seems natural that there would be discontinuities in the velocity, and this would in turn bring in discontinuities in the matter distribution. We shall, in the present paper, therefore consider a spherically symmetric universe in which the collapsing region is separated from a background (which for simplicity we take a homogeneous universe of the Friedman type) by an empty region.

Our model is thus similar to that of Einstein and Straus (1945), except that, instead of their singularity at the spatial origin, we have a collapsing dust distribution over a finite region about the origin.

2. Metrics in the three regions and transformation formulae

For the collapsing region we can use a cosmological line element, so that we have

$$ds^2 = dt^2 - \frac{e^q}{(1+Zr^2/4R_1{}^2)^2}(dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\phi^2) \tag{1}$$

for $r \leqslant a$, where q is a monotonically decreasing function of t, $Z = 0$, $+1$ or -1 and R_1 is a constant.

The field equations give

$$0 = -\frac{Ze^{-q}}{R_1{}^2} - \frac{\partial^2 q}{\partial t^2} - \frac{3}{4}\left(\frac{\partial q}{\partial t}\right)^2 \tag{2}$$

$$8\pi\rho_1 = \frac{3Ze^{-q}}{R_1{}^2} + \frac{3}{4}\left(\frac{\partial q}{\partial t}\right)^2 \tag{3}$$

ρ_1 being the matter density in the collapsing region, which is assumed to be uniform.

For $r > a$ and extending to the homogeneous expanding universe we assume an empty space, where we have the Schwarzschild line element

$$ds^2 = \left(1-\frac{2m}{R}\right) dT^2 - \left(1-\frac{2m}{R}\right)^{-1} dR^2 - R^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{4}$$

Lastly, beyond the empty-space region we have the expanding universe, where the line element is

$$ds^2 = d\tau^2 - \frac{e^g}{(1+K\rho^2/4R_2^{\,2})^2}(d\rho^2 + \rho^2\, d\theta^2 + \rho^2 \sin^2\theta\, d\phi^2) \tag{5}$$

for $\rho \geqslant b$, where g is a monotonically increasing function of τ, $K = 0$, $+1$ or -1 and R_2 is a constant.

The field equations give, analogous to (2) and (3),

$$0 = -\frac{Ke^{-g}}{R_2^{\,2}} - \frac{\partial^2 g}{\partial \tau^2} - \frac{3}{4}\left(\frac{\partial g}{\partial \tau}\right)^2 \tag{6}$$

$$8\pi\rho_2 = \frac{3Ke^{-g}}{R_2^{\,2}} + \frac{3}{4}\left(\frac{\partial g}{\partial \tau}\right)^2 \tag{7}$$

ρ_2 being the uniform matter density in the universe.

There exist transformations to make the line elements (1) and (4) or (4) and (5) continuous. The transformation formulae are for (1) and (4) (Raychaudhuri 1953)

$$\left.\begin{aligned} R &= \xi^2 \\ dT &= \frac{8r(\partial\xi/\partial r)_t(\partial\xi/\partial t)_r}{(\partial q/\partial t)(\xi^2-2m)}\,dt + \frac{\xi^6\,\partial q/\partial t}{2r(\xi^2-2m)}\,dr \end{aligned}\right\} \tag{8}$$

where $\xi = \xi(r, t)$ satisfies the differential equation

$$\left(\frac{\partial \xi}{\partial x}\right)^2 = \tfrac{1}{4}\xi^2 + \tfrac{1}{16}\xi^6\left(\frac{\partial q}{\partial t}\right)^2 - \tfrac{1}{2}m \tag{9}$$

in which $x = \ln r$, and the Schwarzschild mass m is related to the density ρ_1 by

$$m = \tfrac{4}{3}\pi\rho_1 \frac{a^3 e^{3q/2}}{(1+Za^2/4R_1^{\,2})^3}. \tag{10}$$

ξ satisfies, to ensure the continuity at $r = a$, the boundary condition

$$\xi^4(a, t) = \frac{a^2 e^q}{(1+Za^2/4R_1^{\,2})^2}. \tag{11}$$

The quantity ξ can be expressed in terms of elliptic functions. In his paper Raychaudhuri (1953) showed that the transformation (8) makes the first derivatives of the metric tensor components also continuous. Again, for the continuity of (4) and (5) we have relations similar to equations (8), (9), (10) and (11):

$$\left.\begin{aligned} R &= \eta^2 \\ dT &= \frac{8\rho(\partial\eta/\partial\rho)_\tau(\partial\eta/\partial\tau)_\rho}{(\partial g/\partial\tau)(\eta^2-2m)}\,d\tau + \frac{\eta^6\,\partial g/\partial\tau}{2\rho(\eta^2-2m)}\,d\rho \end{aligned}\right\} \tag{8'}$$

$$\left(\frac{\partial \eta}{\partial y}\right)^2 = \tfrac{1}{4}\eta^2 + \tfrac{1}{16}\eta^6\left(\frac{\partial g}{\partial \tau}\right)^2 - \tfrac{1}{2}m \tag{9'}$$

in which $y = \ln \rho$ and

$$m = \tfrac{4}{3}\pi\rho_2 \frac{b^3\, e^{3g/2}}{(1+Kb^2/4R_2{}^2)^3} \tag{10'}$$

ρ_2 being the density in the universe and b the boundary value of the radial variable ρ. Also

$$\eta^4(b,\tau) = \frac{b^2\, e^g}{(1+Kb^2/4R_2{}^2)^2}. \tag{11'}$$

Equations (10) and (10′) establish a relation between the collapsing region and the universe in such a way that the mass of the matter in the collapsing region may be said to be equal to the mass that would have been present in the region $0 < \rho < b$ if the universe were completely homogeneous.

We may next examine the relation between the flux of time at the boundary $r = a$ and the flux of time for the observer in the universe. Comparing (8) and (8′), we have

$$\xi = \eta$$

or

$$\left(\frac{\partial\xi}{\partial t}\right)_r dt + \left(\frac{\partial\xi}{\partial r}\right)_t dr = \left(\frac{\partial\eta}{\partial\tau}\right)_\rho d\tau + \left(\frac{\partial\eta}{\partial\rho}\right)_\tau d\rho \tag{12}$$

and

$$\frac{8r(\partial\xi/\partial r)_t(\partial\xi/\partial t)_r}{\partial q/\partial t}\,dt + \frac{\xi^6\,\partial q/\partial t}{2r}\,dr = \frac{8\rho(\partial\eta/\partial\rho)_\tau(\partial\eta/\partial\tau)_\rho}{\partial g/\partial\tau}\,d\tau + \frac{\eta^6\,\partial g/\partial\tau}{2\rho}\,d\rho. \tag{13}$$

For equations (12) and (13) we now obtain

$$\left\{\frac{8\rho(\partial\eta/\partial\rho)_\tau(\partial\eta/\partial\tau)_\rho}{\partial g/\partial\tau} - \frac{\eta^6(\partial g/\partial\tau)(\partial\eta/\partial\tau)_\rho}{2\rho(\partial\eta/\partial\rho)_\tau}\right\} d\tau = \left\{\frac{8r(\partial\xi/\partial r)_t(\partial\xi/\partial t)_r}{\partial q/\partial t} - \frac{\eta^6(\partial g/\partial\tau)(\partial\xi/\partial t)_r}{2\rho(\partial\eta/\partial\rho)_\tau}\right\} dt + \left\{\frac{\xi^6\,\partial q/\partial t}{2r} - \frac{\eta^6(\partial g/\partial\tau)(\partial\xi/\partial r)_t}{2\rho(\partial\eta/\partial\rho)_\tau}\right\} dr.$$

Remembering that at the boundary $r = a$, $dr = 0$, we obtain

$$d\tau = \frac{(\partial\xi/\partial t)_r\,\partial g/\partial\tau}{(\partial\eta/\partial\tau)_\rho\,\partial q/\partial t}\left\{\frac{16r\rho(\partial\xi/\partial r)_t(\partial\eta/\partial\rho)_\tau - \eta^6(\partial g/\partial\tau)\,\partial q/\partial t}{16\rho^2(\partial\eta/\partial\rho)_\tau{}^2 - \eta^6(\partial g/\partial\tau)^2}\right\} dt. \tag{14}$$

Now, as the boundary of the collapsing region approaches the Schwarzschild singularity $R = 2m$, $\eta^2 = \xi^2 \to 2m$, and from equation (9′) we find that the denominator of the right-hand side of equation (14) tends to vanish. Now, if $K = 0$ or -1, $\partial g/\partial\tau$ will be positive, and hence with $\partial q/\partial t$ negative the numerator could not vanish. Thus any interval of time dt on the surface of the collapsing system would correspond to an infinite time for the expanding universe. However, if $K = +1$ the universe would after a finite time interval start contracting, so that dt may correspond to a finite interval of time in the universe. Indeed, signals would actually reach the universe after it too had collapsed inside the Schwarzschild singularity.

3. Development of event horizon

One has for a radial light ray in the Schwarzschild coordinate system

$$\int_{T_1}^{T_2} dT = \int_{R_1}^{R_2} \frac{dR}{1-2m/R} \tag{15}$$

an equation connecting the time T_1, at which light leaves the source at the boundary $R = R_1 = \xi_a{}^2$ of the collapsing system, with time T_2, at which it reaches the observer at the boundary $R = R_2 = \eta_b^2$ of the expanding system. Integrating equation (15), we obtain

$$T_2 - T_1 = R_2 - R_1 + 2m \ln \frac{R_2 - 2m}{R_1 - 2m}$$

$$= \eta_b^2 - \xi_a^2 + 2m \ln \frac{\eta_b^2 - 2m}{\xi_a^2 - 2m}$$

so that from equations (11) and (11′) we have

$$T_2 - \frac{b\, e^{g/2}}{1 + Kb^2/4R_2^2} - 2m \ln \left(\frac{b\, e^{g/2}}{1 + Kb^2/4R_2^2} - 2m \right)$$

$$= T_1 - \frac{a\, e^{q/2}}{1 + Za^2/4R_1^2} - 2m \ln \left(\frac{a\, e^{q/2}}{1 + Za^2/4R_1^2} - 2m \right). \quad (16)$$

Now, if $K = 0$, we have from equations (7) and (10′)

$$e^{g/2} = \alpha T_2^{2/3}$$

where

$$\alpha = \frac{3}{4} \left(\frac{8m}{b} \right)^{1/2} > 0$$

so that the left-hand side of equation (16) is unbounded for large values of T_2. Hence, to an observer at $\rho = b$ in the expanding system, no event horizon develops until the collapsing system reaches the Schwarzschild singularity. However, when $K = -1$, for large values of T_2, we have

$$e^{g/2} \simeq T_2/R_2$$

so that, if

$$\frac{b/R_2}{1 - b^2/4R_2^2} > 1$$

i.e. if

$$b > 2(\sqrt{2} - 1)R_2 \quad (16')$$

the left-hand side of equation (16) becomes negative, so that the collapsing zone will be cut off even before the Schwarzschild singularity is reached. But, for a suitable value of the boundary value b of ρ, no event horizon develops before the Schwarzschild singularity is reached by the collapsing system. Thus equation (16′) sets a condition on the boundary value b.

4. Calculation of frequency shift

From equations (8) and (15) we have

$$\int_{T_1}^{T_2} dT = \int_{\xi_a}^{\eta_b} \frac{2\xi\, d\xi}{1 - 2m/\xi^2}. \quad (17)$$

Differentiating (17) with respect to time T_1, we have

$$\frac{dT_2}{dT_1} - 1 = \frac{2\eta_b(\partial\eta/\partial T_2)_b}{1 - 2m/\eta_b^2} \frac{\partial T_2}{\partial T_1} - \frac{2\xi_a(\partial\xi/\partial T_1)_a}{1 - 2m/\xi_a^2}$$

or

$$\left\{ 1 - \frac{2\eta_b(\partial\eta/\partial T_2)_b}{1 - 2m/\eta_b^2} \right\} \frac{dT_2}{dT_1} = 1 - \frac{2\xi_a(\partial\xi/\partial T_1)_a}{1 - 2m/\xi_a^2}. \quad (18)$$

Using equation (8), and remembering that $dr = 0$ at the boundary, we obtain

$$\left(\frac{\partial \xi}{\partial T_1}\right)_a = \frac{(\partial q/\partial t)_a(\xi_a^2 - 2m)}{8(\xi_x)_a} \tag{19}$$

where

$$\xi_x = \frac{\partial \xi}{\partial x}.$$

Using equation (8′), we have an analogous expression

$$\left(\frac{\partial \eta}{\partial T_2}\right)_b = \frac{(\partial g/\partial \tau)_b(\eta_b^2 - 2m)}{8(\eta_y)_b} \tag{19′}$$

where

$$\eta_y = \frac{\partial \eta}{\partial y}.$$

Hence from equations (18), (19) and (19′) we have

$$\frac{\partial T_2}{\partial T_1} = \frac{1 - \xi_a^3(\partial q/\partial t)_a/4(\xi_x)_a}{1 - \eta_b^3(\partial g/\partial \tau)_b/4(\eta_y)_b}. \tag{20}$$

Again, the proper times at reception and emission corresponding to dT_2 and dT_1 are, respectively,

$$ds_2 = \left[\left(1 - \frac{2m}{\eta_b^2}\right)\left\{1 - \frac{\eta_b^6(\partial g/\partial \tau)^2}{16(\eta_y)_b^2}\right\}\right]^{1/2} dT_2 \tag{21}$$

$$ds_1 = \left[\left(1 - \frac{2m}{\xi_a^2}\right)\left(1 - \frac{\xi_a^6(\partial q/\partial t)^2}{16(\xi_x)_a^2}\right)\right]^{1/2} dT_1. \tag{22}$$

Combining equations (20), (21) and (22) and equating the ratio of the proper periods of the received and emitted light to the ratio of the corresponding wavelengths, we obtain an expression for the wavelength change:

$$\frac{\lambda + d\lambda}{\lambda} = \frac{ds_2}{ds_1} = \left(\frac{1 - 2m/\eta_b^2}{1 - 2m/\xi_a^2}\right)^{1/2} \left\{\frac{(1+\alpha)(1-\beta)}{(1-\alpha)(1+\beta)}\right\}^{1/2} \tag{23}$$

where we have written

$$\left.\begin{aligned} \alpha &= \eta_b^3 \left(\frac{\partial g}{\partial \tau}\right)_b \frac{1}{4(\eta_y)_b} \\ \beta &= \xi_a^3 \left(\frac{\partial q}{\partial t}\right)_a \frac{1}{4(\xi_x)_a} \end{aligned}\right\}. \tag{24}$$

The first term corresponds to the usual gravitational shift. The terms involving α and β indicate the effects of the motions of the observer and the source, respectively. Since the source is in the collapsing system and the observer is in the expanding system, $\partial q/\partial t$ is negative, while $\partial g/\partial \tau$ is positive. Hence α and β are positive and negative, respectively, so that both terms contribute to the red shift.

The expressions for α and β in terms of the mass m, η_b, ξ_a, b, a, R_2 and R_1 are complicated. We shall, however, calculate the effects for the cases $K = Z = 0$ and $K = Z = \pm 1$ separately.

Case (i). $K = Z = 0$

From equations (1), (3), (9) and (11) one has

$$(\xi_x)_a = \tfrac{1}{2}\xi_a \quad \text{and} \quad \xi_a^3 \left(\frac{\partial q}{\partial t}\right)_a = \mp (8m)^{1/2}.$$

The upper sign before the radical corresponds to negative $\partial q/\partial t$ (collapse), while the lower sign corresponds to positive $\partial q/\partial t$ (expansion). Again, from equations (5), (7), (9′) and (11′) one has the analogous expression

$$(\eta_y)_b = \tfrac{1}{2}\eta_b \quad \text{and} \quad \eta_b{}^3 \left(\frac{\partial g}{\partial \tau}\right)_b = \pm (8m)^{1/2}.$$

In this case the upper sign corresponds to positive $\partial g/\partial \tau$ (expansion) and the lower sign to negative $\partial g/\partial \tau$ (contraction).

Hence from equations (23) and (24) one has

$$\frac{\lambda + d\lambda}{\lambda} = \frac{1 \pm (2m)^{1/2}/\eta_b}{1 \mp (2m)^{1/2}/\xi_a}. \tag{25}$$

Equation (25) shows the cut-off for the collapsing system at the Schwarzschild singularity $\xi_a{}^2 = 2m$, since $\eta_b{}^2 > 2m$.

Case (ii). $K = Z = \pm 1$

With the aid of equations (1), (3), (4), (11) and (5), (7), (9′) and (11′) one can reduce equation (23), considering the conditions of continuity of ξ and η, to the form

$$\frac{\lambda + d\lambda}{\lambda} = \left(\frac{1 - 2m/\eta_b{}^2}{1 - 2m/\xi_a{}^2}\right)^{1/2} \left\{\frac{1 \mp \lambda_1 (2m/\eta_b{}^2)^{1/2}}{1 \pm \lambda_1 (2m/\eta_b{}^2)^{1/2}}\right\}^{1/2} \left\{\frac{1 \pm \lambda_2 (2m/\xi_a{}^2)^{1/2}}{1 \mp \lambda_2 (2m/\xi_a{}^2)^{1/2}}\right\}^{1/2} \tag{26}$$

where

$$\lambda_1 = 1 + \left(1 - \frac{\eta_b{}^2}{2m}\right) \frac{Kb^2}{R_2{}^2} \left(1 - \frac{Kb^2}{4R_2{}^2}\right)^2 \tag{27}$$

$$\lambda_2 = 1 + \left(1 - \frac{\xi_a{}^2}{2m}\right) \frac{Za^2}{R_1{}^2} \left(1 - \frac{Za^2}{4R_1{}^2}\right)^2. \tag{28}$$

It is interesting to note the implication of equations (25) and (26) as the boundary approaches the Schwarzschild singularity, i.e. $\xi_a{}^2 \to 2m$. For the ever-expanding universe ($K = 0, -1$) $\eta_b{}^2 > 2m$, and hence both (25) and (26) give a cut-off corresponding to an infinite $d\lambda/\lambda$. For the model $K = +1$, however, eventually a contracting phase ensues, so $\eta_b{}^2$ goes on decreasing indefinitely. However, as $\eta_b{}^2 < 2m$ (Schwarzschild singularity) λ_1 also becomes greater than unity; the possibility then exists that the light signal from the collapsed system would reach the boundary of the universe when $\eta_b{}^2 < 2m$.

5. Conclusion

Whereas in Raychaudhuri's paper it was found that the collapsing zone is excluded from the onset of contraction, we find here that for $K = 0$, and also for a suitable value of b for $K = -1$, the contraction may be observed up to the development of the Schwarzschild singularity.

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